Building Risk-Optimal Portfolio Using Evolutionary Strategies

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Abstract. In this paper, an evolutionary approach to portfolio optimization is proposed. In the approach, various risk measures are introduced instead of the classic risk measure defined by variance. In order to build the risk-optimal portfolio, three evolutionary algorithms based on evolution strategies are proposed. Evaluations of the approach is performed on financial time series from the Warsaw Stock Exchange.

Keywords: Evolutionary Computation, Evolution Strategies, Portfolio Optimization, Risk Measures, Financial Time Series, Warsaw Stock Exchange.

1 Introduction

Evolutionary Algorithms were successfully incorporated into many fields of science and technology, among other things, into economics and finance ([4], [6], [8], [12]). This paper presents another application of Evolutionary Algorithms in this domain, namely an evolutionary approach to the problem of portfolio optimization, which consists in minimizing the risk of an investment for a desired level of expected return.

Although some analytical methods are well-known for classic versions of the problem ([1], [3]), an extension of the problem by introducing more complex risk measures and loosing several artificial assumptions requires a new efficient approach, which cannot be developed on the basis of classic methods due to the irregularity of the objective function and the search space. However, the opportunities provided by Evolutionary Algorithms ([2], [10]) may lead to an efficient optimization of portfolio structures.

Moreover, apart from theoretical constraints, which are usually considered in financial models, the approach presented focus also on a few practical constraints such as budget constraints, which means that the user of the system has only finite amount of money, as well as investor capabilities and preferences, which means that the user has to obey commonly used regulations such as paying transaction fees. Moreover, an important constraint is constituted by time restrictions and hardware limits.

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The paper is structured in following manner: First, in Section 2, the exact problem definition is given. Section 3 describes the real-life data from the Warsaw Stock Exchange used in the computations. In Section 4, the proposed approach to the problem of portfolio optimization is presented in detail. Section 5 justifies the proposed approach presenting several benchmarks and experiments. Finally, Section 6 concludes the paper and points possible future extensions.

2 Problem Definition

In this paper, we focus on the main goal of investors, which is to optimally allocate their capital among various financial assets. Searching for an optimal portfolio of stocks, characterized by random future returns, seems to be a difficult task and is usually formalized as a risk-minimization problem under a constraint of expected portfolio return. The risk of portfolio is often measured as the variance of returns, but many other risk criteria have been proposed in the financial literature ([1]). Portfolio theory may be traced back to the Markowitz's seminal paper ([9]) and is presented in an elegant way in [3].

Consider a financial market on which n risky assets are traded. Let

$$\mathbf{R} = (R_1, R_2, \ldots, R_n)'$$

be the square-integrable random vector of random variables representing their return rates.

Denote as $\mathbf{r} = (r_1, r_2, \ldots, r_n)' \in \mathbb{R}^n$ the vector of their expected return rates

$$\mathbf{r} = (\mathbf{E}[R_1], \mathbf{E}[R_2], \ldots, \mathbf{E}[R_n])' = \mathbf{E}[\mathbf{R}]$$

and as $\mathbf{V}$ the corresponding covariance matrix which is assumed positive definite.

A portfolio is a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)' \in \mathbb{R}^n$ verifying

$$x_1 + x_2 + \ldots + x_n = 1. \tag{1}$$

Hence x_i is the proportion of capital invested in the i-th asset.

Denote as X the set of all portfolios. For each portfolio $\mathbf{x} \in X$, we define

$$R_{\mathbf{x}} = x_1 R_1 + x_2 R_2 + \ldots + x_n R_n = \mathbf{x}'\mathbf{R}$$

as the random variable representing the portfolio return rate and then

$$\mathbf{E}[R_{\mathbf{x}}] = x_1 r_1 + x_2 r_2 + \ldots + x_n r_n = \mathbf{x}'\mathbf{r}$$

is the portfolio expected return rate.

For a fixed level $e \in \mathbb{R}$ of expected return rate, let

$$X_e = \{\mathbf{x} \in X : \mathbf{E}[R_{\mathbf{x}}] = e\}$$

be the set of all portfolios leading to the desired expected return rate e. Therefore, the classic Markowitz's problem of portfolio optimization may be formulated as finding $\tilde{\mathbf{x}} \in X_e$ such that:

$$\mathbf{Var}[R_{\tilde{\mathbf{x}}}] = \min\{\mathbf{Var}[R_{\mathbf{x}}] : \mathbf{x} \in X_e\}, \tag{2}$$

where the variance is considered as the risk measure.

Such a problem, defined in the classic portfolio theory, may be solved using analytical methods ([3]). The approach has very strong mathematical foundations and completed theoretical models. In spite of this, there is a lot of competitive practical approaches, which extend these theoretical models to real investment market.

Dealing with theoretical models, strong assumptions should be fulfilled. Most of them are completely artificial and unreal such as some of the classic assumptions. Unfortunately, loosing these assumptions, the model becomes more and more complex, hence it cannot be solved in classic way.

In spite of its wide diffusion in the professional and academic worlds, the classic model is often criticized for its artificial assumptions. Although it is an interesting theoretical model, its practical applications may often misfire. Competitive portfolio optimization methods base on heuristic descended from empirical observations. Also the artificial intelligence is often used to optimize a stocks portfolio ([5], [8]).

In this paper, we extend the classic model by introducing several alternative risk measures instead of variance. The extended model cannot be solved by analytical methods, because of its complexity and lack of proper optimization tools. However, evolutionary algorithms presented in the next section can do with this problem returning satisfying results.

In order to extend the classic problem of portfolio optimization, let replace the criteria (2) by the criteria

$$\varrho(\tilde{\mathbf{x}}) = \min\{\varrho(\mathbf{x}) : \mathbf{x} \in X_e\}, \tag{3}$$

where $\varrho : \mathbf{X} \to \mathbb{R}$ is a risk measure, i.e. a function which assign to each portfolio $\mathbf{x} \in X$ its risk $\varrho(\mathbf{x}) \in \mathbb{R}$. For instance, it may represent semivariance of the return rate

$$\varrho(\mathbf{x}) = \mathbf{SVar}[\mathbf{R_x}] = \mathbf{E}[(\mathbf{R_x} - \mathbf{r_x})_-^2],$$

where

$$(\mathbf{R_x} - \mathbf{r_x})_- = \begin{cases} 0, & \text{if } \mathbf{r_x} \le \mathbf{R_x} \\ \mathbf{R_x} - \mathbf{r_x}, & \text{if } \mathbf{R_x} < \mathbf{r_x} \end{cases},$$

the downside risk of the return rate ([1]) or other risk measure, such as these studied in ([7]).

3 Data Description

In practice, all the computations are performed on real-life data from the Warsaw Stock Exchange consisting of financial time series of price quotations of about 40 different stocks over a specific time period. On the basis of these data, the return rates are estimated. Let

$$(\xi_k^{(1)}), (\xi_k^{(2)}), \ldots, (\xi_k^{(n)})$$

denote time series representing prices of stocks $A_1, A_2, \ldots, A_n$ respectively, i.e. for each $i = 1, 2, \ldots, n$ the sequence

$$\xi_0^{(i)}, \xi_2^{(i)}, \ldots, \xi_m^{(i)}$$

contains prices of the stock A_i in consecutive time instants of the specific period and $m+1$ denotes the length of the time period (the same for all the stocks A_i). Further, let

$$r_1^{(i)}, r_2^{(i)}, \ldots, r_m^{(i)}$$

denote a time series of return rates of the stock A_i in consecutive time instants of the specific period, i.e.

$$r_j^{(i)} = \frac{\xi_j^{(i)} - \xi_{j-1}^{(i)}}{\xi_{j-1}^{(i)}}, \quad \text{for } j = 1, 2, \ldots, m.$$

Therefore, for each $i = 1, 2, \ldots, n$, the expected return rate $r_i = \mathbf{E}[R_i]$ and the variance $\mathbf{Var}[R_i]$ may be computed respectively as

$$r_i = \frac{1}{m}\sum_{j=1}^{m} r_j^{(i)}, \quad \mathbf{Var}[R_i] = \frac{1}{m-1}\sum_{j=1}^{m}(r_j^{(i)} - r_i)^2.$$

Similarly, one may compute the correlation matrix $\mathbf{\Sigma}$.

For any portfolio $\mathbf{x} \in \mathbb{R}^n$, the expected return rate $r_{\mathbf{x}} = \mathbf{E}[\mathbf{R_x}]$ and the semivariance $\mathbf{SVar}[\mathbf{R_x}]$ may be computed respectively as

$$r_{\mathbf{x}} = \sum_{i=1}^{n} x_i r_i, \quad \mathbf{SVar}[\mathbf{R_x}] = \frac{1}{m-1}\sum_{j=1}^{m}(\sum_{i=1}^{n} x_i r_j^{(i)} - r_{\mathbf{x}})^2.$$

In our approach, the problem of portfolio optimization is expressed as finding a vector $\mathbf{x} \in \mathbb{R}^n$ minimizing a given risk measure $\varrho : \mathbf{X} \to \mathbb{R}$ under the constraint that the expected return rate $r_{\mathbf{x}}$ is no lower than a given value $e \in \mathbb{R}$ (the value e is often given as the expected return rate of a specific initial portfolio $\mathbf{x}_0$). It is considered in the context of a given set of stocks $A_1, A_2, \ldots, A_n$ with time series of its prices over a given time period. Such a problem with irregular risk measures constitutes hard optimization problem and is studied with evolutionary algorithms.

4 Evolutionary Algorithms

In our research, we focus on solving the optimization problem defined in the previous section using evolution strategies and their modifications. Three different algorithms are applied: a simple evolution strategy with the famous Rechenberg's *1/5 success rule* (called **ES1**) ([2], [10]), a classic $ES(\mu, \lambda)$ evolution strategy with mutation parameters encoded in individuals (called **ES2**) ([2], [10]) and a more advanced $ES(\mu, \lambda, \varrho, \kappa)$ evolution strategy with mutation by multidimensional rotations (called **ES3**) ([11]). Modifications in these algorithms concern fitness evaluation, some restrictions in recombination and choosing the next population.

4.1 Search Space and Objective Function

In all the algorithms, the portfolio is encoded as an n-dimensional real number vector, where n is the number of stocks in the portfolio under consideration. The search space is the entire n-dimensional real number space $\mathbb{R}^n$ – although some elements of the space may not represent portfolios, they may be normalized to fulfill the condition (1). The objective function is basically given by the risk measure $\varrho : \mathbf{X} \rightarrow \mathbb{R}$, but is slightly modified by some heuristic additional factors, which are also considered by certain financial experts and stock market analysts, such as the β coefficients of the portfolio evaluated and the specific initial portfolio. Therefore, the following objective function is studied

$$F(\mathbf{x}) = \frac{1}{1 + \varepsilon_1 \cdot \varrho(\mathbf{x}) + \varepsilon_2 \cdot |\beta_{\mathbf{x}} - \beta_{\mathbf{x}_0}| + \varepsilon_3 \cdot \mathbf{Cov}(R_{\mathbf{x}}, R_{\mathbf{i}})},$$

where $\mathbf{x}_0$ denotes the specific initial portfolio, $R_{\mathbf{i}}$ denotes the return rate of the stock market index and $\beta_{\mathbf{x}}, \beta_{\mathbf{x}_0}$ denote the β coefficients of the portfolio evaluated $\mathbf{x}$ and the specific initial portfolio $\mathbf{x}_0$ respectively. Factors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are used to tune the algorithm and to adjust the importance of each component of the objective function.

These objective functions refer to some heuristics using parameters such as the β coefficient. By introducing the difference between the β_x of the generated portfolio and the β_{x_0} of the portfolio of reference, we penalize the portfolio having β_x far away from β_{x_0} of the reference. Nevertheless, the performance of a solution is defined in terms of expected return and risk of the portfolio over a test period as was mentioned in previous sections.

4.2 Algorithms Initialization

There are used several methods of generating an initial population. The simplest method is random generating with uniform probability. It consists of μ-times random choosing of an individual from the search space, where μ denote the population size. The second method uses an initial portfolio $\mathbf{x}_0$ given by the user. An initial population is chosen from the neighborhood of the given portfolio. It is done by generating a population of random modifications of the initial portfolio.

Every individual in the initial population has to fulfill the financial constraints. Thus, after random generation, every individual undergoes a validation process. If it is not accepted, it is fixed or replaced with other random generated individual. Therefore, the initial population is always correct, which means that fulfills all the desired conditions.

4.3 Evolutionary Operators

In the algorithm, common evolution operators such as reproduction and replacement are used.

In the process of reproduction, a population of size μ generates λ descendants. Each descendant is created from 4 ancestors. Reproduction consists of three parts: parent selection, recombination and mutation, repeated λ times.

Parent selection consists of choosing 4 parents from a population of size μ using one of the most popular methods, so-called "roulette wheel", where the probability of choosing an individual is proportional to its value of the objective function.

Recombination consists of generating one descendant from the 4 parents chosen earlier. It is done by one of two operators chosen randomly with equal probability: either the global intermediary recombination or the local intermediary recombination. In the first operator, genes of the descendant are arithmetic averages of genes of all the 4 parents chosen earlier. In the second operator, 2 parents are chosen from these 4 parents chosen earlier, for each gene separately, and next, the gene of the descendant is the arithmetic average of genes of the 2 parents.

After recombination, a mutation operator is applied. It depends on the algorithm. In the **ES1** algorithm, mutation is controlled by the Rechenberg's *1/5 success rule*: a random noise is added to each gene of the descendant, it is drawn with gaussian distribution $\mathcal{N}(0, \sigma)$, and the parameter σ is increased in each iteration of the algorithm when, in 5 last iterations, the number of mutations leading to improvement of individuals exceeded 20% of total mutations and decreased otherwise. The amount of increase and decrease is fixed.

In the **ES2** algorithm, mutation is controlled by a parameter σ, different for each individual, encoded in an additional chromosome in each individual. The parameter σ is an n-dimensional real number vector, in which each coordinate defines the standard deviation of the gaussian distribution of random noise for each gene of the individual. The parameter σ undergoes the evolution as well.

In the **ES3** algorithm, mutation is more complex. It is controlled by two parameters σ and α, different for each individual, encoded in two additional chromosomes in each individual. These parametres are used to drawn a random direction in the n-dimensional real number space and a random movement in this direction. Details of the mutation may be found in [11].

After mutation, each descendant must undergo a process of verification in order to satisfy the constraints as in the case of generating individuals during the algorithm initialization.

Finally, in the replacement process, a new population of size μ is chosen from the old population of size μ and its λ descendants via the deterministic selection. In the **ES3** algorithm, there are an additional constraint that each individual can survive no more than κ generations in the history.

4.4 Termination Criteria

Termination criteria include several conditions. The first condition is defined by the acceptable level of evaluation function value. The second is based on the homogeneity of population, defined as the minimal difference between the best and the worst portfolio. The third condition is defined as the maximal number of generations. The algorithm stops when one of them is satisfied.

5 Validation of the Approach

In order to validate our approach, a large number of experiments on various financial time series from the Warsaw Stock Exchange were performed. Each financial time series included daily quotations of a given stock.

Each experiment began with choosing stocks $A_1, A_2, \ldots, A_n$ constituting financial instruments available for an investor. Normally, $n = 10$ stocks were randomly chosen among all the stocks in our financial database (consisting of about 40 stocks from the Warsaw Stock Exchange). Next, an initial portfolio $\mathbf{x}_0$ was drawn corresponding to partitioning the investor's capital among the available stocks. Afterwards, a time instant t was chosen and the evolutionary algorithms presented in the previous section was applied to optimise the initial portfolio at the time t, i.e. to find an optimal portfolio $\mathbf{x}$ of equal or higher expected return rate $\mathbf{E}[\mathbf{x}] \geq \mathbf{E}[\mathbf{x}_0]$ and minimum risk measure $\varrho(\mathbf{x})$. All the computations concerning estimation of return rates were done over the period preceding the time instant t, i.e. over the time period $(t - \Delta t, t)$, where usually $\Delta t = 25$.

In each experiment, a few issues were investigated. First, the risk $\varrho(\mathbf{x}_0)$ of the initial portfolio $\mathbf{x}_0$ were compared with the risk $\varrho(\mathbf{x})$ of the built optimal portfolio $\mathbf{x}$ and the risk $\varrho(\mathbf{x}_*)$ of the portfolio $\mathbf{x}_*$ optimal according to the Markowitz model. Naturally, the built portfolio had always lower risk than the initial one. What is interesting, the built portfolio $\mathbf{x}$ had always lower risk than the portfolio $\mathbf{x}_*$, which proved that a portfolio optimal according to variance is not optimal according other risk measures. Second, the three evolutionary algorithms were compared according to the computing time and the quality of solutions. Finally, actual return rates of all the three portfolios, namely $\mathbf{x}_0, \mathbf{x}, \mathbf{x}_*$, were computed over a future time period $(t, t + \Delta t)$, where usually $\Delta t = 25$, with 5 portfolio restructurisations, each after 5 time instants, and compared. Naturally, the stock prices over the future time period were unknown during the process of portfolio optimization. Return rates were compared also with return rates of the so-called Buy&Hold strategy and the stock market index (the Buy&Hold strategy corresponds to investing all the capital at the beginning of the test period and holding to the end of it).

Table 1 shows a summary of the experiments addressing the first issue - risk comparison. Each experiment was repeated 30 times with different parameters. One may see that the risk of the optimised portfolio $\mathbf{x}$ is significantly lower than the risk of the initial portfolio as well as the minimum variance portfolio.

Table 2 presents performances of the three evolutionary algorithms. For each algorithm, a number η of experiments, when the optimum found by the algorithm was better than the optima found by the other two, is shown in the second column. In the third column, the average computing time, for each algorithm, is shown. Naturally, the computing time depends on parameters of the algorithm, which varied in experiments, but due to size constraints of the paper, they are only summarized. Not surprisingly, the last algorithm has the best performance, but it also requires the longer computing time.

Table 3 presents the comparison of return rates of the three portfolios, $\mathbf{x}_0, \mathbf{x}, \mathbf{x}_*$, over a future time period $(t, t + \Delta t)$. In the first, second and third column, there is

Table 1. Risk comparison for the initial portfolio $\mathbf{x}_0$, the built optimal portfolio $\mathbf{x}$, and the portfolio $\mathbf{x}_*$ optimal according to the Markowitz model

n	$\varrho(\mathbf{x}_0)$	$\varrho(\mathbf{x})$	$\varrho(\mathbf{x}_*)$
10	0.9322	0.3743	0.5362
10	0.2873	0.1983	0.3108
10	0.8642	0.6134	0.7134
10	0.8734	0.5654	0.6154
20	0.5481	0.3270	0.4642
20	0.8135	0.6141	0.8242
20	0.7135	0.4035	0.5193
20	0.6135	0.3985	0.4792

Table 2. Performance (a number of experiments, when the optimum found by the algorithm was better than the optima found by the others) and computing time for the three algorithms

Algorithm	η	Computing Time
ES1	57	38 s
ES2	75	53 s
ES3	108	97 s

Table 3. The number of experiments where the initial portfolio $\mathbf{x}_0$, the built optimal portfolio $\mathbf{x}$, and the portfolio $\mathbf{x}_*$ optimal according to the Markowitz model outperformed the others as well as the number of experiments where the portfolio $\mathbf{x}$ outperformed the Buy&Hold strategy and the stock market index

$\mathbf{x}_0$	$\mathbf{x}$	$\mathbf{x}_*$	B&H	Index
0	28	2	30	4
3	23	4	27	3
2	21	7	28	3
0	30	0	30	7
1	27	2	29	5
0	29	1	30	6
1	26	3	29	5
2	24	4	28	4

the number of experiments where the portfolio $\mathbf{x}_0, \mathbf{x}, \mathbf{x}_*$, respectively, turned out to outperform the others. In the next two columns, there are the number of experiments where the portfolio $\mathbf{x}$ outperformed the Buy&Hold strategy and the stock market index, respectively. One may see that the profit obtained by the evolutionary built portfolio and the minimum variance portfolio is usually slightly higher than the profit of the initial portfolio. In order to study that, further research is necessary. However, these results concern more economy and finance than evolutionary computation, so are not discussed in detail in this paper.

6 Conclusions and Perspectives

In this paper, a new approach to portfolio optimization was proposed. It rejects some assumptions used in theoretical models, introduces transaction costs and alternative risk measures such as the semivariance and the downside risk. The approach has been evaluated and validated using real data from the Warsaw Stock Exchange.

In order to evaluate this approach, the obtained investment strategy has been compared with the Buy&Hold strategy and the stock market index. To reduce the time period bias on performance, several time series have been selected. The results have demonstrated that the evolutionary approach is capable of investing more efficiently than the simple Buy&Hold strategy and the stock market index in some cases.

The presented approach can be still improved by modifying evolutionary operators, especially recombination. The fitness function study can also increase the efficiency of the method. Additional effort should be put on methods of portfolio validation in order to eliminate unacceptable solutions at the moment of its creation.

The evolutionary approach in stock trading is still in an experimentation phase. Further research is needed, not only to build a solid theoretical foundation in knowledge discovery applied to financial time series, but also to implement an efficient validation model for real data. The presented approach seems to constitute a practical alternative to classical theoretical models.

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